Subtracting (3) from (4) and transposing,

$$AE \times BC + BE \times BC + AE \times BE = AD \times DC + AD \times BC + DC \times BC. \quad (5)$$

We also have

$$AB : BC = AD : DC,$$

$$AC : BC = AE : EB,$$

$$\therefore \quad BC \times AD = AB \times DC = (AE + BE)DC, \quad (6)$$

$$BC \times AE = AC \times EB = (AD + DC)EB. \quad (7)$$

Substituting these values of $BC \times AD$ and $BC \times AE$ in (5) and dropping $BE \times CD$ from both sides we have

$$AD \times EB + BC \times EB + AE \times EB = AD \times DC + AE \times DC + BC \times DC.$$

Or

$$(AD + BC + AE)EB = (AD + AE + BC)DC.$$

$$\therefore \quad EB = DC.$$

Substituting $EB = DC$ in (5) we get, after dropping $EB \times BC$ from both sides,

$$AE \times BC + AE \times BE = AD \times EB + AD \times BC$$

or

$$(BC + BE)AE = (BC + BE)AD.$$

Hence

$$AE = AD.$$

Therefore

$$AB = AC.$$

III. Concerning Huntington's *Continuum and Other Types of Serial Order*.[1]

By Lester S. Hill, Princeton University.

If A_1 is any subclass of an ordered class A, then all elements a of A which are ordinally less than every element of A_1 ($a < A_1$) constitute a lower segment of A; and all elements $a > A_1$ constitute an end segment. If A_1 and A_2 are two subclasses with $A_1 < A_2$ then all elements a for which $A_1 < a < A_2$ constitute a mid segment. Of course A_1 or A_2, or each of them, may reduce to a single element of A.

The order type of A is continuous if

(D) A is dense: between every two elements of A lie other elements;

(G) A is gap-free: If A' is a lower segment, A'' an end segment, and $A = A' + A''$, then A' has a last element or else A'' has a first. ($A = A' + A''$ implies $A' < A''$.)

The ordered class A of real numbers satisfies (D), (G) and also

(F_0) A is denumerably framed: A contains a *denumerable* subclass B which is dense in A; between every two elements of A lie elements of B.

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[1] See Book Review on page 325 of this issue.

The more general continuous types for which Huntington has space to cite examples satisfy (D), (G) and the following weakened form of (F_0),—

(F) A is denumerably framed by segments: on every segment of A lie segments which are denumerably framed.

It may therefore be of interest to illustrate a simple type of continuous order satisfying neither (F_0) nor (F).

§ I. Let $H = \{h\}$ be the class of all numerical sequences of type ω

$$h = (e_1, e_2, e_3, \cdots),$$

where e_n is any real number for every finite index n, only finite ordinal numbers occurring as indices.

Then, $\quad h' = (e_1', e_2', e_3', \cdots) \quad$ and $\quad h'' = (e_1'', e_2'', e_3'', \cdots)$

being two elements of H we define

$h' < h''$ (h' less than h'') if, from some index on, $e_n' < e_n''$,
$h' > h''$ (h' greater than h'') if, " " " " $e_n' > e_n''$,
$h' = h''$ (h' equal to h'') if, " " " " $e_n' = e_n''$.

If one of these three relations holds we call h' and h'' comparable; otherwise h and h' are incomparable.

Observe that if $h' \lesseqgtr h''$ and $h'' \lesseqgtr h'''$ then $h' \lesseqgtr h'''$.

§ II. We omit the proof that there exist subclasses $K = \{k\}$ of H, every k being an h, with the properties of closure:

(α) K contains no pair of incomparable, and no pair of equal sequences.

(β) The class $H - K$ contains no sequence which is comparable with every k but equal to no k.

§ III. $K = \{k\}$, which is, to to speak, a maximum (closed) subclass of serially ordered sequences in H, has the striking ordinal property:

If K' and K'' are denumerable subclasses of K such that $K' < K''$, then K contains an infinite subclass K''' such that $K' < K''' < K''$.

For, since K' and K'' are denumerable, we can arrange them

$$K' = (k_1'k_2'k_3'\cdots), \qquad K'' = (k_1''k_2''k_3''\cdots),$$

k_1' being the first sequence of K' in this arrangement, etc. We have, more in detail,

$$k_1' = (k_{11}', k_{12}', k_{13}', \cdots); \qquad k_1'' = (k_{11}'', k_{12}'', k_{13}'', \cdots)$$

$$k_2' = (k_{21}', k_{22}', k_{23}', \cdots); \qquad k_2'' = (k_{21}'', k_{22}'', k_{23}'', \cdots)$$

. . . .

and, of course, every k_n' is less than every k_m'' $(K' < K'')$.

We can now determine a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that

$$k_{1n}' < k_{1n}'' \qquad (n \geqq n_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}',\ k_{2n}',\ \cdots,\ k_{mn}' < k_{1n}'',\ k_{2n}'',\ \cdots,\ k_{mn}'' \qquad (n \geqq n_m)$$

and, obviously, the sequence $h = (e_1, e_2, e_3, \cdots)$ where the e_n are any real numbers such that

$$k_{1n}' < e_n < k_{1n}'' \qquad (n_1 \leqq n < n_2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}',\ k_{2n}',\ \cdots,\ k_{mn}' < e_n < k_{1n}'',\ k_{2n}'',\ \cdots,\ k_{mn}'' \ (n_m \leqq n < n_{m+1})$$

is an element of H greater than K' and less than K''. Therefore, if we suppose the mid segment of K determined by $K' < K''$ to be zero (that is, without elements), it is clear that h is comparable with every element of K, and is itself an element of K; and we have a contradiction. In this argument we use the observation made at the end of § I.

If K' or K'', or each of these classes, reduces to a finite class or to a single element of K, the reader may supply the appropriate argument and obtain the same conclusion as above. Hence K is certainly dense, satisfying (D).

§ IV. Hausdorff defined K and proved that its potency is that of the real number continuum.

We need not trouble to ascertain whether K satisfies condition (G). For we can obtain from K very easily a class J ordered according to (D) and (G), and having the property discussed in § III.

Let $J = \{j\}$ be the class of all those lower segments in K which are without last element; and define $j_1 < j_2$ if the segment j_2 contains elements of K not occurring in j_1. A very hasty reading of Huntington's book will enable the student to see at once that J, thus ordered, satisfies (D) and (G); and that it contains a subclass L which is dense in J and similar (ordinally equivalent) to K: namely, the class of lower segments in K corresponding to partitions $K = K' + K''$ $(K' < K'')$, in which K' has no last element, but K'' has a first.

The density of L in J and the ordinal equivalence of L and K furnish an easy proof (based on that property of K discussed in § III) for the statement:

J contains no denumerably framed segment. While continuous it is widely divergent from the continuous types cited in the manual under review. It has a property meaning more than the mere absence of the property (F) which characterizes those types.

The reader may again follow indications by Hausdorff in the construction of number systems "Gröszensysteme" which are serially ordered, but non-Archimedean, and otherwise different from the familiar system of real numbers.

Let $H = \{h\}$ be, as before, the class of all real number sequences; and, if $h' = (e_1', e_2', e_3', \cdots)$, $h'' = (e_1'', e_2'', e_3'', \cdots)$ are two elements of H define:

$$h' \pm h'' = (e_1' \pm e_1'', \quad e_2' \pm e_2'', \ \cdots)$$

$$h' \gtrless h'' \text{ for } e_m' = e_m'' \ (m < n) \text{ and } e_n' \gtrless e_n''.$$

H is herewith furnished with serial order; and with an associative and commutative addition having the property: $h_1 + h_2 > h_1 + h_3$ if $h_2 > h_3$. The symbols mh, h/n, mh/n where m, n are positive or negative integers have an obvious meaning; and H contains a definite element described by $r_1h_1 + r_2h_2 + \cdots + r_nh_n$ where $h_1 \cdots h_n$ are any elements of H and $r_1r_2 \cdots r_n$ are any rational *real* numbers.

Like the Gröszensystem upon which Veronese bases his geometry of the intuitional continuum, H is not Archimedean; if $h_1 < h_2$ are an arbitrary pair of its elements, we can not assert that for some positive integer n, $nh_1 > h_2$; in fact such an assertion would be false in the obvious instance

$$h_1 = (0, 1, 0, 0, \cdots), \qquad h_2 = (1, 0, 0, 0, \cdots).$$

H contains relative "infinites" and "infinitesimals" of all finite orders; in striking contrast with the real number system, in which no constant number is infinitesimal or infinite with respect to any other constant.

Desiring a continuously ordered Gröszensystem we proceed again to the class $J = \{j\}$ of all lower segments in H without last element; and define $j_1 < j_2$ when the segment j_2 contains elements of H not occurring in j_1. As before, we obtain a continuously ordered J (but of order type very different from that of the former J) containing a subclass L which is dense in J and ordinally equivalent to H. That the extension of H to J is possible appears in the obvious fact that H is dense; that an extension is necessary is shown by the simple partition $H = H' + H''$ $(H' < H'')$ where H' contains all elements of H ordinally less than some $(0, e, 0, 0, \cdots)$, H'' all elements ordinally greater than some $(e, 0, 0, 0, \cdots)$ as e assumes all real values > 0: a partition which explodes condition (G).

The means of defining suitably an operation of addition in J is suggested by Dedekind's procedure in extending the class of rational numbers to the class of real numbers (cf. C. Jordan, *Cours d'Analyse*, Vol. I). Let $j_1 < j_2$ be elements of J representing the lower segments H_1 and H_2 of H. We define $j_1 + j_2$ as that lower segment (clearly without last element) in H which consists of all sums $h_1 + h_2$ of an element in H_1 and an element in H_2.

The properties of addition as defined in H validate this form of definition; and establish the desirable relation: if $l_1 < l_2$ of L correspond to $h_1 < h_2$ of H, then $l_1 + l_2$ is an element of L and corresponds to $h_1 + h_2$ of H. We have in L a class which is equivalent to H with respect to order and also with respect to addition. Since L is thus completely isomorphic with H, and is moreover dense in J, we are more than enabled to argue that J, like H, is non-Archimedean; indeed the Archimedean property involves only a relation between the ordering and the adding in a system with order and addition.

Hausdorff has given a general analysis of a large class of Gröszensysteme with order and addition which are non-Archimedean. He defined the particular case H explicitly but is not responsible for shortcomings which may be found in J.